SOME RESULTS ON BOREL STRUCTURES WITH APPLICATIONS TO SUBSERIES CONVERGENCE IN ABELIAN TOPOLOGICAL GROUPS

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ABSTRACT

We show that in an abelian topological group subseries convergence depends only on the Borel field generated by the topology. We also prove a result about measurability of a limit for a pointwise converging sequence of measurable mappings into an analytic topological space.

In the first part of this paper we make an improvement of a result due to the second author (see [2], th. 2, p. 246). In the second part we prove a theorem about measurability of a limit function. We use this result which may also have some interest in itself. In the third part we show that the results may be considerably improved if we restrict ourselves to the category of analytic abelian topological groups. In both the first and the third part we obtain improvements of results due to Kalton, which was obtained by a different method (see [5]).

We consider the Cantor group $K = \{0,1\}^{\tilde{N}}$. Equipped with the usual product topology and product group structure K is a compact metrizable abelian topological group. It may be convenient to identify in the natural way elements of K with subsets of $\tilde{N}$. A mapping ϕ from K into an abelian group G is said to be finitely additive if $\phi(x+y) = \phi(x) + \phi(y)$ for all disjoint x and y in K (that means for all x and y fulfilling $xy = 0$). For $a \in K$, we use the notation K_a for the compact subgroup of K consisting of those elements x in K for which $xa = x$. $K_0 \subseteq K$ is the countable subgroup of K consisting of all elements of finite support. For the n'th coordinate of an element x in K we write $x(n)$, e_m is the element in K for which $e_m(n) = \delta_{mn}$, and e the element for which $e(n) = 1$ for all n.

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A subset $S \subseteq G$ of an abelian topological group $(G, +, \mathcal{O})$ is called σ-bounded if for every neighbourhood U in G, S may be covered with countably many translates of U.

If $(X, \mathcal{O})$ is a Hausdorff topological space then a subset $A \subseteq X$ has the Baire property (by definition) if there exist an open set $U \subseteq X$ such that the set $A \triangleleft U = (A \setminus U) \cup (U \setminus A)$ is of the first category in X (contained in a countable union of closed sets with empty interior). The system of sets with the Baire property forms the σ-field of BP-measurable sets (containing of course the Borel sets).

THEOREM 1. *Let $(G, +, \mathcal{O})$ be an abelian Hausdorff topological group and $\phi : K \to G$ a finitely additive mapping. Suppose either that ϕ is Borel measurable or alternatively suppose that $\phi(K)$ is a σ-bounded subset of G and for any $a \in K$ the restriction of ϕ to K_a is relatively BP-measurable. Then for any x in K, we have:*

$$\phi(x) = \sum_{n=1}^{\infty} x(n)\phi(e_n)$$

where the sum converges uniformly for $x \in K$ with respect to the uniform structure on G induced by the group operation.

PROOF. First we assume that the group G is separable and metrizable. Obviously we may as well assume that d is an invariant complete metric on G generating the topology. We assume furthemore that ϕ has the weaker measurability property stated in the theorem.

First we show that the mapping

$$\psi(x) = \lim_{p \to \infty} \sum_{n=1}^{p} x(n)\phi(e_n)$$

is well defined since the limit exists uniformly in $x \in K$ with respect to the uniform structure induced by the group operation. Suppose this is not the case. Then it is easy to see that there exist an $e > 0$ and a sequence $y_i \in K_0$ of disjoint elements such that $d(\phi(y_i), 0) \geqq e$ for all $i \in \tilde{N}$. As the topology of G has a countable base and ϕ is BP-measurable we can find a dense G_δ set A in K such that the restriction of ϕ to A is continuous. Since K_0 is countable we may assume that A is invariant under translations with elements in K_0. Let $x \in A$ and put

$$x_i' = x + y + xy_i$$

and

$$x_i'' = x + xy_i$$

(the addition with respect to the group structure in K). Since A is invariant under translations with elements in K_0 both x_i' and x_i'' belongs to A. Clearly we have $x_i' \to x$ and $x_i'' \to x$ and therefore $\phi(x_i') - \phi(x_i'') \to 0$ in G for $i \to \infty$, but x_i'' and y_i are disjoint with sum x_i' and therefore

$$d(\phi(x_i') - \phi(x_i''), 0) = d(\phi(y_i), 0) \geqq e\,.$$

This contradiction shows that ψ is indeed well defined in the above sense. Clearly ψ is continuous and finitely additive. Hence the mapping $\phi'(x) = \phi(x) - \psi(x)$ is BP-measurable and finitely additive and furthermore $\phi'(e_n) = 0$ for all n. It follows that $\phi'(x) = \phi'(y)$ for all x and y in K for which $\{n \in \tilde{N} \mid x(n) \neq y(n)\}$ is finite. The topological zero-one law (see [2], th. 1, p. 246; note that this law is valid also for mapping with values in a Hausdorff space with the second axiom of countability as follows immediately from the proof) now shows that $\phi'(x)$ is equal to a constant $g \in G$ on a dense G_δ subset A of K. By choosing $x, y, z \in A$ with $yz = 0$ and $x = y + z$ (see [2], lemma, p. 247) it is seen that $g = 0$. Since also $(e + A)$ is a dense G_δ set, $A \cap (e + A)$ is non empty, and so there is $x, y \in A$ with $x + y = e$ (with respect to the group operation on K). This means that x and y are disjoint with union e. From this we conclude $\phi'(e) = 0$. We want to show that $\phi'(x) = 0$ for all $x \in K$. For a fixed x in K let us define $\phi_x(y) = \phi'(xy)$. As the mapping $y \to xy$ from K to K_x is open and continuous it is easy to see that it is $(BP(K), BP(K_x))$ measurable, hence $\phi_x : K \to G$ is BP-measurable. A similar argument as above now yields $\phi_x(e) = \phi'(x) = 0$. Hence ϕ' vanishes, and this concludes the proof in this special case where G is separable and metrizable.

It is clearly enough to obtain the conclusion of the theorem for any weaker topology $\mathscr{P}$ on G which makes G into a metrizable topological group (for any $\mathcal{O}$-neighbourhood V in G there exists a topology $\mathscr{P}$ on G with these properties for which V is a $\mathscr{P}$-neighbourhood). By a suitable division we may even assume that $\mathscr{P}$ is Hausdorff. The second part of the theorem is now easily reduced to the case already proved, as a σ-bounded subset of a metrizable group is separable.

Suppose now that ϕ is Borel measurable. We reduce to the metrizable case as before. From [3] it follows that $\phi(K)$ is automatically separable in G. Then we have reduced to the case already proved. Note that the result of [3] does not depend on any "pathological" set-theoretical axioms but just the axiom of choice (compactness arguments) and the usual (naive) set theory not including the continuum hypothesis.

COROLLARY. *Subseries convergence in an abelian topological group depends only on the Borel structure generated by the topology.*

PROOF. This follows immediately from the theorem.

Let $(X,\mathscr{B})$ be a measurable space and $(Y,\mathscr{O})$ a Hausdorff topological space. Let $f_n : X \to Y$ be a sequence of $(\mathscr{B},\mathscr{B}(Y))$ measurable functions. Suppose that the limit point $f_\infty(x) = \lim_{n\to\infty} f_n(x)$ exists in Y for all $x \in X$. What can we say about the measurability properties of the limit function $f_\infty : X \to Y$? It is easy to show that if the space $(Y,\mathscr{O})$ is regular and fully Lindelöf then f is $(\mathscr{B},\mathscr{B}(Y))$ measurable. It is not known whether this need to be the case if $(Y,\mathscr{O})$ is an analytic Hausdorff topological space.

If $\mathscr{B}$ is a Borel structure on a set X then the paving $CS(\mathscr{B})$ is the system of all complements of sets which are Souslin sets with respect to $\mathscr{B}$. The paving $S(\mathscr{B}) \cap CS(\mathscr{B})$ is a Borel structure which in general is strictly finer then $\mathscr{B}$ but coincides with $\mathscr{B}$ if $\mathscr{B}$ is the Borel structure of an analytic Hausdorff space (a smooth Borel structure).

THEOREM 2. *Let the situation be as above. If the space $(Y,\mathscr{O})$ is an analytical Hausdorff space then f_∞ is measurable with respect to the Borel field $\mathscr{B}(Y)$ on Y and the σ-field $S(\mathscr{B}) \cap CS(\mathscr{B})$ on X.*

PROOF. Let Δ denote the diagonal in $Y \times Y$. As Y is Hausdorff, for all $y \in (Y \times Y)\setminus\Delta$ there exist open sets O_y and U_y in Y for which $y \in O_y \times U_y$ and $(O_y \times U_y) \cap \Delta = \emptyset$. By using the fully Lindelöf property of $Y \times Y$ (which is analytic) we now conclude that there exists a sequence O_n, U_n of open sets in Y such that $\cup_n O_n \times U_n = (Y \times Y)\setminus\Delta$. Hence the set of finite intersections of sets in the sequences O_n and U_n in a countable base P_n for a Hausdorff topology $\mathscr{P}$ coarser than the topology $\mathscr{O}$. In the following we consider Y with this topology $\mathscr{P}$. As $\mathscr{P} \subseteq \mathscr{O}$ we still have $f_n \to f$ pointwise, and because of the Borel isomorphism theorem ([4], co. II 5.3 p. 83) $\mathscr{P}$ and $\mathscr{O}$ generates the same Borel field $\mathscr{B}(Y)$.

We now use the "Cantor group technique" to describe the function f_∞. Let us define the function $x \to k_x$ from X to K by:

$$k\,(n) = 1 \Leftrightarrow \exists m_0 \in \tilde{N} : m \geqq m_0 \Rightarrow f_m(x) \notin P_n\,.$$

The function k is $(\mathscr{B},\mathscr{B}(K))$ measurable. This follows easily from the measurability of the f_m's and the equality:

$$\{x\in X \mid k_x(n) = 1\} = \bigcup_{q=1}^{\infty} \bigcap_{m=q}^{\infty} f_m^{-1}(P_n^c)$$

(c denotes set-theoretical complement). Since k is measurable the inverse image by k of any coanalytic set in K belongs to $CS(\mathscr{B})$. The equality:

$$\left(\bigcup_{n=1}^{\infty} k_x(n)P_n\right)^c = \{f_\infty(x)\}$$

is easily proved. Now let B be a Borel set in Y. Then we have:

$$f_\infty^{-1}(B) = \{x\in X \mid \{f_\infty(x)\} \subseteq B\} = \{x \in X \mid (\bigcup_{n=1}^{\infty} k_x(n)P_n)^c \subseteq B\}$$

$$= \{x\in X \mid B^c \setminus (\bigcup_{n=1}^{\infty} k_x(n)P_n) \neq \emptyset\}^c = k^{-1}(S_1^c)$$

where S_1 is the projection on the first coordinate of the set:

$$S = \{(k,y)\in K \times Y \mid y \in B^c \setminus (\bigcup_{n=1}^{\infty} k(n)P_n)\}$$

$$= \{(k,y) \mid y\in B^c\} \cap \{(k,y) \mid y \in \bigcup_{n=1}^{\infty} k(n)P_n\}^c .$$

As the intersection of a Borel set and a closed set S is a Borel set in $K \times Y$ and therefore S_1^c is coanalytic in K, hence $f_\infty^{-1}(B) = k^{-1}(S_1^c) \in CS(\mathscr{B})$. As also B^c is a Borel set in Y we have $f^{-1}(B^c) \in CS(\mathscr{B})$, that is we have $f^{-1}(B) \in CS(\mathscr{B}) \cap S(\mathscr{B})$ which finishes the proof.

We consider an abelian group $(G,+)$. Suppose that $\mathscr{P}$ is any Hausdorff topology on G and $g_n (n \in \tilde{N})$ is a sequence of elements in G. The series $\sum_{n=1}^{\infty} g_n$ is called $\mathscr{P}$-subseries convergent if $\lim_{q\to\infty} \sum_{n=1}^{q} x(n)g_n$ exists with respect to $\mathscr{P}$-topology for all x in K. $\mathscr{P}$ is called a group topology if $(G,\mathscr{P})$ is a topological group, and it is called translation invariant if all translations are homeomorphisms with respect to $\mathscr{P}$-topology.

The following theorem is an improvement of a result of N. J. Kalton (see [5] th. 3, p. 407).

THEOREM 3. *Let $\mathscr{P}$ and $\mathscr{O}$ be two topologies on G. Suppose $\mathscr{P} \subseteq \mathscr{O}$, $\mathscr{O}$ is an analytic group topology and $\mathscr{P}$ is translation invariant and Hausdorff. Then $\mathscr{P}$-subseries convergence is equivalent with $\mathscr{O}$-subseries convergence.*

PROOF. Suppose Σg_n is $\mathscr{P}$-subseries convergent. We define a mapping $\phi: K \to G$ by

$$\phi(x) = \mathscr{P} - \lim_{q \to \infty} \sum_{n=1}^{r} x(n) g_n = \mathscr{P} - \lim_{q \to \infty} S_q(x).$$

For every $q \in \tilde{N}$ the mapping $s_q : K \to G$ is continuous (for $\mathscr{P}$ or $\mathscr{O}$) because it only takes a finite number of values on sets which are open and closed. From Theorem 2 we conclude that ϕ is Borel measurable with respect to the Borel structure generated by the $\mathscr{P}$ topology which equals the Borel structure generated by the $\mathscr{O}$ topology since both are smooth (note that the Borel structure on K is smooth). By using that $\mathscr{P}$ is translation invariant it is easily seen that for $x \in K_0$, $y \in K$ and $xy = 0$ we have $\phi(x + y) = \phi(x) + \phi(y)$. By inspection of the proof of Theorem 1 it is seen that the measurability of ϕ and the preceding kind of additivity of ϕ is enough to ensure that there exist a dense G_δ set $A \subseteq K$ which is invariant under translations with elements from K_0 such that the restriction of ϕ to A is continuous with respect to $\mathscr{O}$-topology, (note that the Borel structure of $\mathscr{P}$ and $\mathscr{O}$ coincide). Moreover we may conclude that $\Sigma x(n) g_n$ is a uniform $\mathscr{O}$-Cauchy series (uniform for $x \in K$). If we choose $x, y, z \in A$ with $yz = 0$ and $y + z = x$ (see the proof of Theorem 1) and put $y_q(n) = y(n)$ for $n = 1, \cdots q$, and $y_q(n) = 0$ for all other n we have $y_q + z \in A$ and $y_q + z \to x$ for $q \to \infty$, hence

$$\phi(y_q + z) = \phi(y_q) + \phi(z) = s_q(y) + \phi(z) \to \phi(x)$$

for $\mathscr{O}$-topology. So we see that there is an element $y \in A$ for which $\mathscr{O} - \lim_{q \to \infty} s_q(y)$ exists in G and as the topology $\mathscr{P}$ is Hausdorff this limit must be $\phi(y)$. We want to show that $\phi(x) = \sum_{n=1}^{\infty} x(n) g_n$ for all $x \in$ with respect to $\mathscr{O}$-topology. As in the proof of Theorem 1, it is seen that it is enough to consider the case where G is separable and metrizable and then to complete G to $\bar{G}$. We may define a mapping $\psi : K \to \bar{G}$ by:

$$\psi(x) = \mathscr{O} - \lim_{q \to \infty} s_q(x).$$

Since ψ is continuous the mapping $\phi' = \phi - \psi$ is Borel measurable and has the same additivity properties as ϕ and in the same way as in the proof of Theorem 1 we see that we may assume that $\phi'(x) = g$ for all $x \in A$. By choosing an $y \in A$ for which $\mathscr{O} - \lim_{q \to \infty} s_q(x) = \phi(y)$ we get $\phi(y) = \psi(y)$ that is $g = 0$. Now the proof can be finished exactly as the proof of Theorem 1.

It may very well be that the above result could be obtained without the assumption that $\mathscr{P}$ is translation invariant. This is open at the time of writing.

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